Exercise 34

If R denotes the reaction of the body to some stimulus of strength x, the *sensitivity* S is defined to be the rate of change of the reaction with respect to x. A particular example is that when the brightness x of a light source is increased, the eye reacts by decreasing the area R of the pupil. The experimental formula

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}$$

has been used to model the dependence of R on x when R is measured in square millimeters and x is measured in appropriate units of brightness.

(a) Find the sensitivity.

(b) Illustrate part (a) by graphing both R and S as functions of x. Comment on the values of R and S at low levels of brightness. Is this what you would expect?

Solution

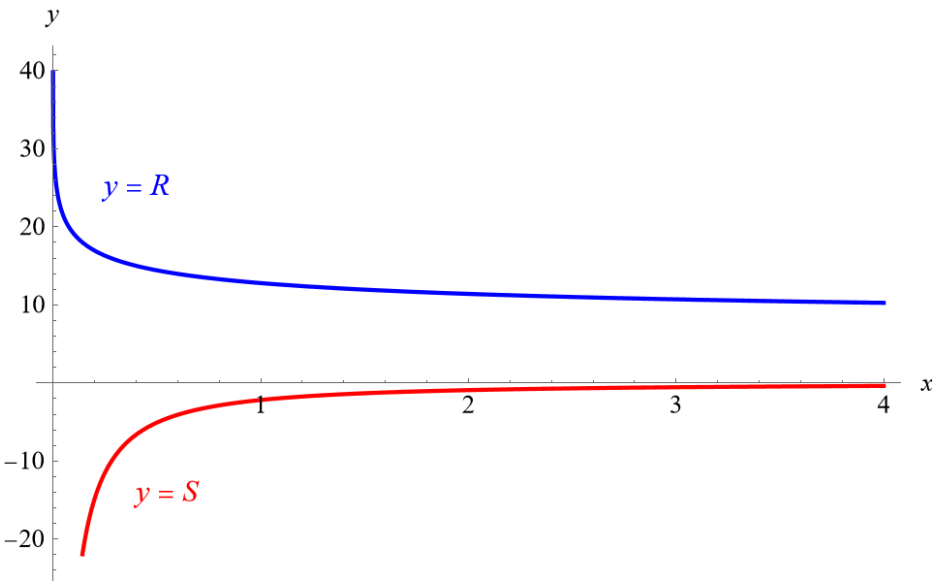
Part (a)

Take the derivative of R to get S by using the quotient rule.

$$\begin{aligned}
S &= \frac{dR}{dx} \\
&= \frac{d}{dx}\left(\frac{40 + 24x^{0.4}}{1 + 4x^{0.4}}\right) \\
&= \frac{\left[\dfrac{d}{dx}(40 + 24x^{0.4})\right](1 + 4x^{0.4}) - \left[\dfrac{d}{dx}(1 + 4x^{0.4})\right](40 + 24x^{0.4})}{(1 + 4x^{0.4})^2} \\
&= \frac{[24(0.4x^{-0.6})](1 + 4x^{0.4}) - [4(0.4x^{-0.6})](40 + 24x^{0.4})}{(1 + 4x^{0.4})^2} \\
&= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1 + 4x^{0.4})^2} \\
&= \frac{-54.4x^{-0.6}}{(1 + 4x^{0.4})^2} \\
&= -\frac{54.4}{x^{0.6}(1 + 4x^{0.4})^2}
\end{aligned}$$

Part (b)

Below is a graph of the reaction and sensitivity versus the stimulus.



At low levels of brightness the reaction drops sharply from 40 to about 20. In the limit as $x \to \infty$,

$$\lim_{x\to\infty} R = \lim_{x\to\infty} \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}} = \lim_{x\to\infty} \frac{40x^{-0.4} + 24}{x^{-0.4} + 4} = \frac{24}{4} = 6.$$

In the limit as $x \to 0^+$, the sensitivity becomes infinitely negative,

$$\lim_{x\to 0^+} S = \lim_{x\to 0^+} -\frac{54.4}{x^{0.6}(1 + 4x^{0.4})^2} = -\frac{54.4}{0^+} = -\infty,$$

which doesn't seem to be realistic.